Problem 7

In the following problems, find the limit of the given sequence as $n \to \infty$.

$$(1 + n^2)^{1/\ln n}$$

Solution

Take the limit as $n \to \infty$, using l'Hôpital's rule where it's appropriate.

$$
\begin{aligned}
\lim_{n \to \infty} (1 + n^2)^{1/\ln n} &= \lim_{n \to \infty} \exp\left[\ln(1 + n^2)^{1/\ln n}\right] \\[2mm]
&= \lim_{n \to \infty} \exp\left[\frac{1}{\ln n} \ln(1 + n^2)\right] \\[2mm]
&= \exp\left[\lim_{n \to \infty} \frac{\ln(1 + n^2)}{\ln n}\right] \\[2mm]
&\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \exp\left[\lim_{n \to \infty} \frac{\frac{d}{dn}[\ln(1 + n^2)]}{\frac{d}{dn}(\ln n)}\right] \\[2mm]
&= \exp\left(\lim_{n \to \infty} \frac{\frac{1}{1+n^2} \cdot 2n}{\frac{1}{n}}\right) \\[2mm]
&= \exp\left(\lim_{n \to \infty} \frac{2n^2}{1 + n^2}\right) \\[2mm]
&= \exp\left(\lim_{n \to \infty} \frac{2}{\frac{1}{n^2} + 1}\right) \\[2mm]
&= \exp\left(\frac{2}{0 + 1}\right) \\[2mm]
&= e^2
\end{aligned}
$$